**CALCULUS OF VARIATIONS AND OPTIMIZATION METHODS**

# Part I. Variations calculus

## Lecture 7. Bolza problem

We have the method of the analysis for the problem of the minimization of the integral functional, which depends from one or many unknown functions of one or many variables. The functional can depend from derivatives of unknown functions of arbitrary order. These problems can be transformed to differential equations. It can be solved with given boundary conditions. However there exist a lot of practical problems without fixed conditions on the boundary of the given set. We will try to extend the previous results to these problems. We will consider the river crossing problem as an example.

### 7.1. Problem statement

We considered before the problems with given values on the boundary of the set only. However there exist extremum problems without boundary conditions. The minimizing functional can depends of the value of the boundary state also in this case. Consider the integral



where *F* and *G* are smooth enough given functions.

**Problem 7.1**. *Find the function v*, *which minimizes the functional I.*

**Definition 7.1**. *The Problem* 7.1 *is called* ***Bolza problem***.

### 7.2. Necessary conditions of extremum

We will use the standard technique. Let *u* be a solution of Bolza problem. Determine the function



where *σ* is the number, and *h* is an arbitrary smooth enough function on the interval .

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| **Question**: *What is the boundary condition for the function h?* |

We do not any boundary conditions for the function *h* because the function  is admissible without any suppositions.

We have the equality  for all *σ*, so the function *f* has the minimum at zero. Find the derivative of *f* at zero.

**Lemma 7.1**. *The derivative of the function f at the zero is*

 (7.1)

**Proof**. Determine the value



Using Taylor formula, we find





where  as  So we get



After division by *σ* and passing to the limit as  we get



After integration by parts we have



Therefore the previous equality can be transformed to (7.1).

**Theorem 7.1**. *If the smooth enough function u is the solution of Bolza problem, then it satisfies Euler equation*

 (7.2)

*with boundary conditions*

 (7.3)

** (7.4)

**Definition 7.2**. *The equalities* (7.3), (7.4) *are called* ***transversality conditions***.

**Proof**. We have the equality

 (7.5)

for all function *h*. It is true for the function *h*, which is equal to zero on the boundary too. So we obtain



Using Basic Lemma of Variations Calculus we have Euler equation (7.2). So the equality (7.5) is transformed to

 (7.6)

The values  and  are arbitrary here. Determine  so we obtain the equality (7.3). Determine  at the equality (7.6), then the condition (7.4) is true. €

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| **Conclusion**: *The Bolza Problem is transformed to the second order differential equation* (*Euler equation*) *with two boundary conditions* (*transversality conditions*). |

Hence Bolza problem can be solved as Lagrange problem with change the given boundary conditions by transversality conditions.

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| **Question**: *What we do if there exists one boundary condition in the problem statement*? |

If we have the mixed case with boundary condition on the one ends of the given interval only, then we obtain Euler equation with two boundary conditions also. One of them is the given boundary condition as in Lagrange problem. Second of them is transversality condition as Bolza problem.

### 7.3. Example

We have the problem of minimization of the integral



Determine



We have Euler equation (7.2):



Its general solution is



where  are constants. Determine transversality conditions (7.3) and (7.4):



Put it in the previous formula; we find



So Euler equation with transversality conditions have the solution  It is the solution of our problem is really because the integral is non-negative, and it is equal to zero only for zero value of the function.

### 7.4. River crossing problem

Consider an application of Bolza Problem. We have the river with line bank and the width *L*. We analyze the movement of the boat from the one bank to the other one. Choose the coordinate system *x*,*y* (see Figure 7.1). The velocity of the river  is known. The velocity of the boat *u* is a constant. It is greater than the maximal velocity of the river. We would like to choose the law of the boat’s movement  such that the time of crossing is minimal.



Figure 7.1. The movement of the boat.

Let the point of start be the origin of the coordinate. So we have the boundary condition

 (7.7)

The velocities of the boat with respect to the given coordinates are described by the differential equations

 (7.8)

where *α* is the angle between the direction of the boat’s movement and the axis *x.* Then we get



Consider the last equality as the equation with respect to the angle *α*. We obtain



So we have



Denote



We obtain the square equation with respect to *z*



Find its solution



The sign “minus” here gives the negative value of the angle. We do not obtain the movement to other bank in this case. So we have



Using first equality (7.7) we find



If we choose some law of the movement during the time from zero to *T*, then the boat move with respect to the coordinate *x* from zero to *L*. After integration we determine the time of crossing

 (7.9)

Hence we would like to find the function , which minimize the time  on the set of the function satisfying the boundary condition (7.7). We do not have second boundary condition because the point of the boat’s finish is unknown. So the right end of the curve is free.

The solution of this problem can be found from Euler equation (7.2), given boundary condition (7.7) and transversality condition (7.4). We have

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and the function *G* is equal to zero. The function *F* does not depend from unknown function in our case. So Euler equation can be transform to the equality

Using the absence of the dependence of the function *F* from *y*, we get Euler equation



Then we have

. (7.10)

Using transversality condition (7.4) we have

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So the constant  is equal to zero. Then we have first order differential equation



We get



We find



After integration with using the boundary condition (7.7) we have



It will be the law of movement of the boat, which realizes the minimal time of crossing.

We can consider the partial case with zero value of the velocity of the river. The value *v* is zero for this case. So  because of the last formula. Therefore we have the movement across the river. Put this result to the formula (7.9). So the time of the movement is equal to  that is the ratio of the way *L* to the velocity *u*.

### Outcome

* The solution of Bolza problem satisfies Euler equation with two transversality conditions.
* If we have the problem with one fixed boundary condition, then the necessary condition of minimum involves Euler equation with given boundary condition and transversality condition at the other boundary.
* The river crossing problem is an example of this theory.

### Task. Minimization of functionals without boundary conditions

**Variants 1-7**. Find the function, which satisfies the boundary condition  and minimize the integral

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|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| variant |  |  |  |  |  |
| 1 |  | *v* | 0 | 1 | 0 |
| 2 |  | -*v* | -1 | 0 | 1 |
| 3 |  | -2*v* | 0 | 1 | 0 |
| 4 |  | 2*v* | -1 | 0 | 1 |
| 5 |  | -3*v* | 1 | 0 | 0 |
| 6 |  | 3*v* | 0 | 1 | 1 |
| 7 |  | -*v* | 0 | 1 | 0 |

**Variants 9-14**. Find the function, which satisfies the boundary condition  and minimize the integral

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|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| variant |  |  |  |  |  |
| 8 |  | *v* | 0 | 1 | 0 |
| 9 |  | -*v* | -1 | 0 | 1 |
| 10 |  | -2*v* | 0 | 1 | 0 |
| 11 |  | 2*v* | -1 | 0 | 1 |
| 12 |  | -3*v* | 1 | 0 | 0 |
| 13 |  | 3*v* | 0 | 1 | 1 |
| 14 |  | -*v* | 0 | 1 | 0 |

**Variants 15-21**. Find the function  which minimize the integral



|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| variant |  |  |  |  |
| 15 |  |  | 0 | 1 |
| 16 |  |  | -1 | 0 |
| 17 |  |  | 0 | 1 |
| 18 |  |  | 0 | 1 |
| 19 |  |  | 1 | 0 |
| 20 |  |  | 0 | 1 |
| 21 |  |  | 0 | 1 |

Steps of the task.

1. Determine Euler equation, and find its general solution.
2. Find the solution of this equation, which satisfies the following boundary conditions: two transversality conditions for the variants 15-21 or one transversality condition and the given boundary condition for other variants.
3. Show the graph of this solution.
4. Calculate the corresponding value of the given integral.
5. Calculate the value of the integral for an arbitrary linear function for the variants 15-21 and linear function, which satisfies given boundary condition for other variants.

**Literature**

1. Алексеев В.М., Тихомиров В.М., Фомин С.В. Оптимальное управление. – М., Наука, 1979. –   
   С. 64-66.
2. Ахиезер Н.И. Лекции по вариационному исчислению. – М., ГИТТЛ, 1956. – С. 105-111.
3. Будылин А.М. Вариационное исчисление. – Санкт-Петербург, СПбГУ, 2001. – [http://www.newlibrary.ru/book/budylin\_a\_m\_/variacionnoe\_ischislenie.html](http://www.newlibrary.ru/book/budylin_a_m_/variacionnoe_ischislenie.html%20)  . – Раздел 5.4.
4. Гельфанд И.М., Фомин С.В. Вариационное исчисление. – М., Физматгиз, 1961. – С. 30-32, 56-63.
5. Гурса Э. Курс математического анализа. Том 3. Часть 2. Интегральные уравнения. Вариационное исчисление. – М.-Л., Гостехиздат, 1934. – С. 222-226.
6. Краснов М.Л., Макаренко Г.И, Киселев А.И. Вариационное исчисление. – М., Наука, 1973. – С. 119-130.
7. Лаврентьев М.А., Люстерник Л.А. Основы вариационного исчисления. Том 2. – М., ОНТИ, 1935. – С. 118-126.
8. Смирнов В.И., Крылов В.И., Канторович Л.В. Вариационное исчисление. – Л., 1933. – С. 47-57.
9. Шарапов В.Г. Руководство по решению задач по курсу «Вариационное исчисление и методы оптимизации». – Волгоград, 2004. – С. 25-29.
10. Эльсгольц Л.Э. Дифференциальные уравнения и вариационное исчисление. – М., Наука, 1969. – С. 327-333.
11. Miersemann E. Calculus of Variations. Lecture Notes. – Leipzig, 2012. – P. 122-129.

### Next step

We have the standard method for solving the problems of minimization for the different integral functionals. These problem involved constraints on the boundary only. However there exist minimization problems with additional constraints. We will try to extend our results to these problems.